HERMITIAN EIGENVALUE PROBLEM AND A NEW PRODUCT IN THE COHOMOLOGY OF FLAG VARIETIES

Shrawan Kumar

May 4, 2004

Let G be a complex semisimple algebraic group and let K be a maximal compact subgroup with their Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Choose a maximal subalgebra (which is necessarily abelian) $\mathfrak{a} \subset \mathfrak{p}$ and let $\mathfrak{a}_+$ be a dominant chamber in $\mathfrak{a}$. Then any $K-$orbit in $\mathfrak{p}$ intersects $\mathfrak{a}_+$ in a unique point.

For any $n \geq 2$, the celebrated *Hermitian eigenvalue problem* concerns determining the following subset Δ_n of $\mathfrak{a}_+^n$:

$$\Delta_n := \{(a_1, \dots, a_n) \in \mathfrak{a}_+^n : \exists (x_1, \dots, x_n) \in \mathfrak{p}^n \text{ with } \sum x_i = 0 \text{ and } x_i \in AdK.a_i\}.$$

By works of several mathematicians including Klyachko, Berenstein-Sjamaar, Belkale, Δ_n is given by certain inequalities parametrized by standard maximal parabolic subgroups P of G and n Schubert cohomology classes $\epsilon_{w_1}^P, \dots, \epsilon_{w_n}^P$ such that

$$\epsilon_{w_1}^P \cdot \dots \cdot \epsilon_{w_n}^P = \epsilon^P,$$

where ϵ^P is the top cohomology class of G/P.

But, as shown by Kumar-Leeb-Millson, these set of inequalities are, in general, not irredundant.

Now, the main topic of this talk is a recent joint work with Belkale. We give a new commutative and associative product in the cohomology $H^*(G/P)$ of any flag variety G/P (which still satisfies the Poincaré duality) and show that the ineqalities determining Δ_n are given in terms of this new product in $H^*(G/P)$ for maximal parabolics P. This results in general in far fewer inequalities determining Δ_n. We show that for simple groups of rank 3, our new set of inequalities is an irredundant system.