Forward Equations for European, Barrier, & American Options in Markovian Models

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The Variance Gamma and Related Financial Models

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Introduction

Consider valuing a European, barrier, or American option on a single stock when the stock price process S is Markovian in itself and time t.

Let $V(S, t)$ be the value of the option at time $t \in [0, T]$ given that $S_t = S$.

Finite differences can be used to numerically solve a boundary value problem producing a matrix approximation $\widehat{V}(S, t)$ for S, t evaluated on some rectangular grid.

Due to issues such as discrete dividends and term structures of interest rates, this is in fact the standard valuation method for these kinds of options.

Why be Backward?

- The standard approach propagates the value function in S backwards in calendar time t.
- For options, one can alternatively propagate the value function in the strike K forward in the maturity variable T.
- This enhances computational efficiency, easing both calibration and marking.
- For barrier options, we presently require independent increments in the log of the underlying price.
- For American options, we impose stationarity and/or independent increments.

Outline

There are 3 parts to this talk:

1. European options: Review of the Dupire forward PDE and its generalization to jumps.

2. Barrier Options: Forward PDE for Knockout Call Values and its generalization to Forward PIDE for Double Knockout Calls.

3. American Options: Four PIDE's for American Put Values.

Markovian Stock Price Process

- No arbitrage $\Rightarrow$ the existence of a risk-neutral probability measure $\mathbb{Q}$ under which all assets return r in expectation.

- Under $\mathbb{Q}$, the stock price S solves the following stochastic differential equation (SDE):

$$\begin{aligned} dS_t &= [r(t) - q(t)]\, S_{t-}dt + \sigma(S_{t-}, t) S_{t-} dW_t \\ &+ \int_{-\infty}^{\infty} S_{t-}(e^x - 1)\, [\mu(dx, dt) - \nu(x, S_{t-}, t) dx dt]\,, \qquad \forall t \in [0, \bar{T}]. \end{aligned}$$

- Thus, the change in the stock price is due to risk-neutral drift, to diffusion, and to jumps.

Markovian Stock Price Process (cont'd)

- Recall that under $\mathbb{Q}$, S solves the following SDE:

$$\begin{aligned} dS_t &= [r(t) - q(t)]\, S_{t-}dt + \sigma(S_{t-}, t)S_{t-}dW_t \\ &\quad + \int_{-\infty}^{\infty} S_{t-}(e^x - 1)\,[\mu(dx, dt) - \nu(x, S_{t-}, t)dxdt]\,, \qquad \forall t \in [0, \bar{T}]. \end{aligned}$$

- The density $\{\nu(x, s, t) : \mathbb{R}, \mathbb{R}^+, [0, \bar{T}] \mapsto \mathbb{R}^+\}$ is used to compensate the jump measure $\mu(dx, dt)$ so that the last term is the increment of a $\mathbb{Q}$ jump martingale.

- Since W is also a $\mathbb{Q}$ martingale, taking risk-neutral expectations and solving the ODE implies:

$$\mathbb{E}^{\mathbb{Q}}[S_t|S_0] = S_0 e^{\int_0^t [r(u) - q(u)]du} \qquad \forall t \in [0, \bar{T}].$$

Part I: European Options and Local Variance

- Assuming no jumps, S solves the following SDE under $\mathbb{Q}$:

$$dS_t = [r(t) - q(t)]\, S_{t-}dt + \sigma(S_{t-}, t)S_{t-}dW_t \qquad \forall t \in [0, \bar{T}].$$

- Let $c(K,T) : \mathbb{R}^+, [0, \bar{T}] \mapsto \mathbb{R}^+$ be the $C^{2,1}$ function denoting market prices of European calls.

- Dupire (1994) shows that the local variance function $\sigma^2(K,T)$ can be directly read from the (assumed smooth) double continuum of call prices, $c(K,T)$:

$$\sigma^2(K,T) = 2\frac{\frac{\partial c}{\partial T}(K,T) + [r(T) - q(T)]\, K\frac{\partial c}{\partial K}(K,T) + q(T)c(K,T)}{K^2\frac{\partial^2 c}{\partial K^2}(K,T)}.$$

The Dupire Forward PDE

- The local vol function arises from solving the following forward PDE for σ: i.e. $\frac{\partial c}{\partial T}(K,T) =$

$$\frac{\sigma^2(K,T)K^2}{2}\frac{\partial^2 c}{\partial K^2}(K,T) - [r(T) - q(T)]\, K\frac{\partial c}{\partial K}(K,T) - q(T)c(K,T).$$

- Given a previously obtained local volatility function, $\sigma(K,T)$, one can append initial and boundary conditions to numerically approximate call prices $\widehat{c}(K,T)$ on a grid of strikes and maturities.

Backward and Forward PIDE for European Calls

- Recall that under $\mathbb{Q}$, S solves the following SDE:

$$dS_t = [r(t) - q(t)]\, S_{t-}dt + a(S_{t-},t)dW_t + \int_{-\infty}^{\infty} S_{t-}(e^x - 1)\,[\mu(dx,dt) - \nu(x,S_{t-},t)dxdt]\,.$$

- For $S > 0, t \in [0,\bar{T}]$, values $V(S,t)$ of path-independent claims solve the backward PIDE: $-\frac{\partial}{\partial t}V(S,t) =$

$$-r(t)V(S,t) + [r(t) - q(t)]S\frac{\partial V}{\partial S}(S,t) + \frac{a^2(S,t)}{2}\frac{\partial^2 V}{\partial S^2}(S,t)$$
$$+ \int_{-\infty}^{\infty}\left[V(Se^x,t) - V(S,t) - \frac{\partial V}{\partial S}(S,t)S(e^x - 1)\right]\nu(x,S,t)dx.$$

- For $K > 0, T \in [0,\bar{T}]$, European call values $c(K,T)$ solve the forward PIDE: $\frac{\partial}{\partial T}c(K,T) =$

$$-q(T)c(K,T) - [r(T) - q(T)]K\frac{\partial c}{\partial K}(K,T) + \frac{a^2(K,T)}{2}\frac{\partial^2 c}{\partial K^2}(K,T)$$
$$+ \int_{-\infty}^{\infty}\left[c(Ke^y,T) - c(K,T) - \frac{\partial c}{\partial K}(K,T)K(e^y - 1)\right]\hat{\nu}(y,K,T)dy.$$

Dual Compensator

- The dual compensating density $\hat{\nu}(y, K, T)$ is implicitly defined in terms of the original compensating density $\nu(x, S, t)$:

$$\int_{-\infty}^{\infty} \left[(Le^x - K)^+ - (L-K)^+ - 1(L > K)L(e^x - 1)\right] \nu(x, L, T)dx$$
$$= \int_{-\infty}^{\infty} \left[(L - Ke^y)^+ - (L-K)^+ + 1(L > K)K(e^y - 1)\right] \hat{\nu}(y, K, T)dy.$$

- The explicit definition is given by $\hat{\nu}(y, K, T) \equiv$:

$$e^{-y}\left[\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial y}\right] \int_{-\infty}^{\infty} \left[\left(e^{x+y} - 1\right)^+ - (e^y - 1)^+ - 1\,(y > 0)\, e^y(e^x - 1)\right] \nu(x, L, T)dx.$$

- The same forward PIDE holds for puts, straddles, and other European path independent claims with a single kink in their payoff.

Part II: Barrier Options

Assuming no jumps, the forward PDE for the Down-and-Out Call value function $D(K,T)$ with $K > H$ is the same as the Dupire forward PDE, i.e. $\frac{\partial D}{\partial T}(K,T) =$

$$\frac{\sigma^2(K,T)K^2}{2}\frac{\partial^2 D}{\partial K^2}(K,T) - [r(T) - q(T)]\, K\frac{\partial D}{\partial K}(K,T) - q(T)D(K,T)$$

with initial condition

$$D(K,0) = (S_0 - K)^+, \text{ for } K > H, \text{ and } S_0 > H.$$

Boundary conditions are

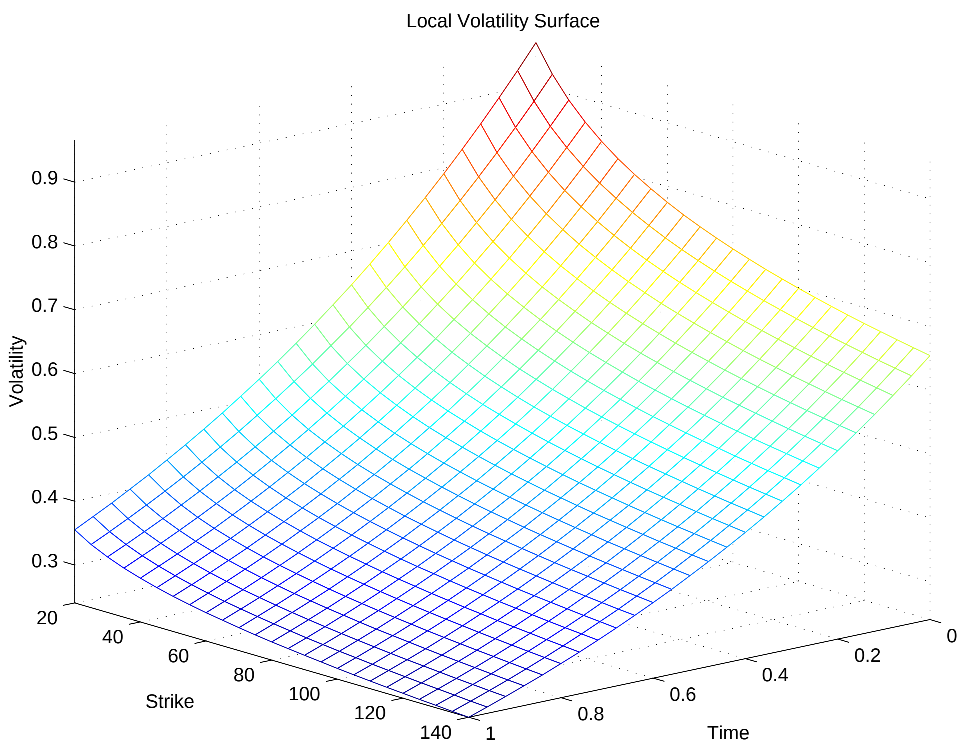
$$D_{KK}(H,T) = 0$$

$$\lim_{K\to\infty} D_{KK}(K,T) = 0$$

Local Volatility Surface

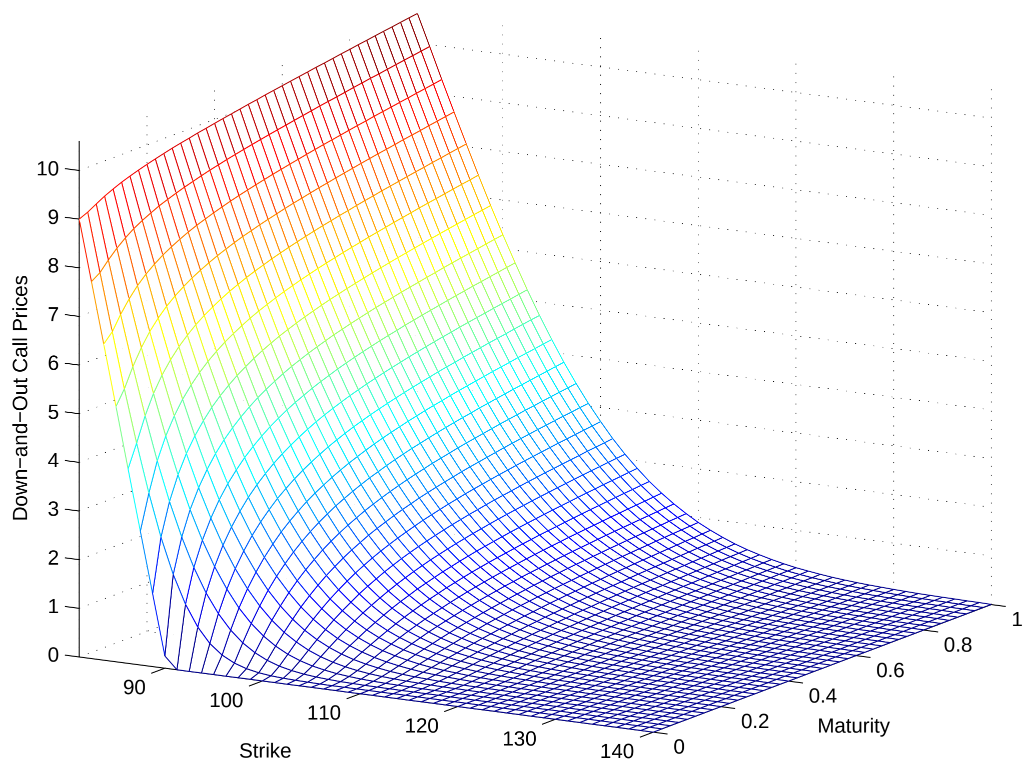
In our numerical examples, we consider the following local volatility surface

$$\sigma(K,T) = 0.7e^{-T}(100/K)^{0.2}$$



Down-and-Out Call Prices

In this illustration, the variables are: barrier $H = 80$, spot $S_0 = 90$, risk-free rate $r = 0.05$, and dividend rate $q = .02$.



Forward PDE for Up-and-Out Calls

- Under no jumps, the forward PDE for UOC is $\frac{\partial U}{\partial T}(K,T) =$

$$\frac{\sigma^2(K,T)K^2}{2}\frac{\partial^2 U}{\partial K^2}(K,T) - [r(T) - q(T)]\, K\frac{\partial U}{\partial K}(K,T) - q(T)U(K,T)$$
$$+(H-K)\frac{\sigma^2(H,T)H^2}{2}\frac{\partial^3 U}{\partial K^3}(H,T), \qquad K < H, T \in [0,\bar{T}],$$

  with initial condition:

$$U(K,0) = (S_0 - K)^+, \text{ for } K < H, \text{ and } S_0 < H.$$

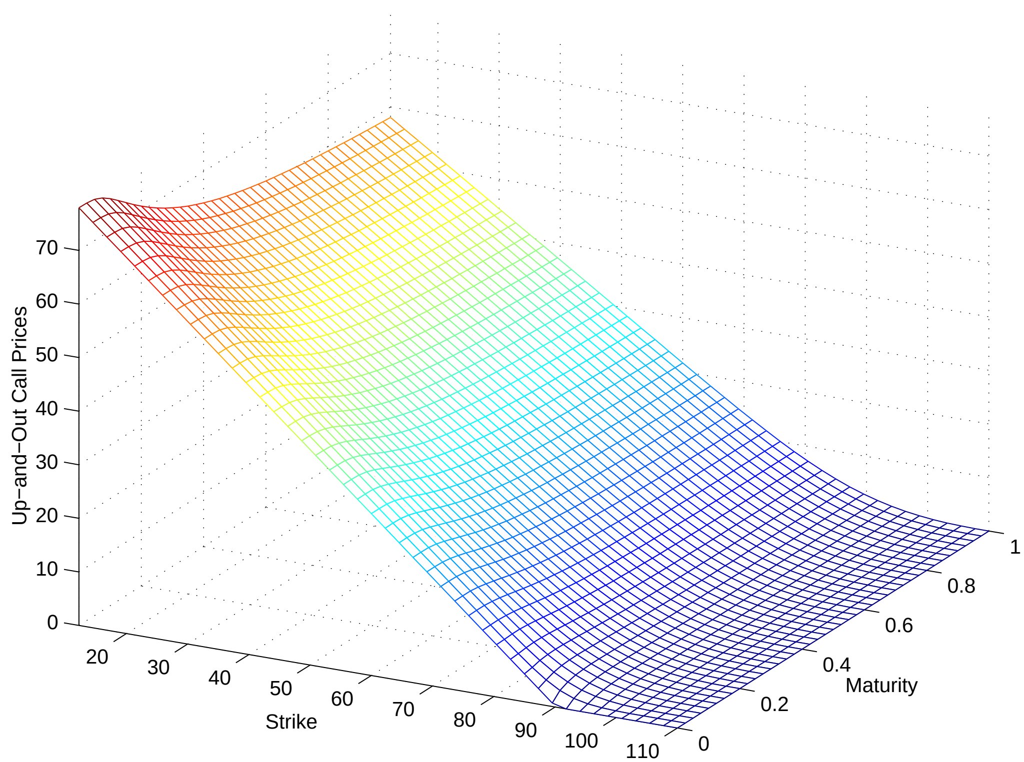
- Boundary conditions are:

$$U_{KK}(0,T) = 0 \text{ and } U_{KK}(H,T) = 0.$$

- The new term on the RHS of the PDE is due to the positive intrinsic value of the vanilla call when $S = H$.

Up-and-Out Call Prices

In this illustration, the variables are: barrier H=110, spot S_0=90, risk-free rate $r = 0.05$, and dividend rate $q = .02$.



Forward PIDE for Double Knockout Calls

Assuming that the jump part of the log price is an an additive process, the following forward PIDE holds for double knockout call values: $\frac{\partial}{\partial T} D^{kc}(K,T) =$

$$-q(T)D^{kc}(K,T) - [r(T)-q(T)]K\frac{\partial}{\partial K}D^{kc}(K,T) + \frac{a^2(K,T)}{2}\frac{\partial^2}{\partial K^2}D^{kc}(K,T)$$

$$+\int_{-\infty}^{\infty}\left[D^{kc}(Ke^{-x},T) - D^{kc}(K,T) - \frac{\partial}{\partial K}D^{kc}(K,T)K(e^{-x}-1)\right]e^x\nu(x,T)dx$$

$$+(H-K)\left[\frac{a^2(H,T)}{2}\frac{\partial^3}{\partial K^3}D^{kc}(H,T) + \int_{0+}^{\infty}\frac{\partial}{\partial K}D^{kc}(He^{-x},T)\nu(x,T)dx\right]$$

$$-\int_{0+}^{\infty}D^{kc}(He^{-x},T)e^x\nu(x,T)dx.$$

Initial and Boundary Conditions

- For a double knockout call, the initial condition is:

$$D^{kc}(K,0) = (S_0 - K)^+, \quad K \in [L,H), S_0 \in (L,H).$$

- Some upper boundary conditions for the double knockout call are that for $T \in [0,\bar{T}]$:

$$D^{kc}(H,T) = \frac{\partial}{\partial K} D^{kc}(H,T) = \frac{\partial}{\partial T} D^{kc}(H,T) = \frac{\partial^2}{\partial K^2} D^{kc}(H,T) = 0.$$

- Likewise, some lower boundary conditions for the double knockout call are that for $T \in [0,\bar{T}]$:

$$D^{kc}(L,T) = \frac{\partial}{\partial K} D^{kc}(L,T) = \frac{\partial}{\partial T} D^{kc}(L,T) = \frac{\partial^2}{\partial K^2} D^{kc}(L,T) = 0.$$

Part III: American Options

When the underlying stock price follows our single factor Markov process, the backward PIDE for pricing American puts is*:

$$\frac{\partial P}{\partial t}(S,t) + \frac{\sigma^2(S,t)S^2}{2}\frac{\partial^2 P}{\partial S^2}(S,t) + [r(t) - q(t)]\, S\frac{\partial P}{\partial S}(S,t) - r(t)P(S,t)$$
$$+ \int_{-\infty}^{+\infty} \left[P(Se^x,t) - P(S,t) - \frac{\partial P}{\partial S}(S,t)S(e^x - 1) \right] \nu(x,S,t)dx$$
$$+\mathbf{1}_{S<S(t)} \left\{ r(t)K_0 - q(t)S - \int_{\ln(S(t)/S)}^{\infty} [P(Se^x,t) - (K_0 - Se^x)]\, \nu(x,S,t)dx \right\} = 0.$$

*For the derivation and numerical solution of the PIDE in the special case of VG, see *Pricing American Options Under Variance Gamma* by Hirsa & Madan

The Backward Free Boundary Problem for American Puts (con'd)

The terminal condition is

$$P(S, T_0) = \max(K_0 - S, 0),$$

and the boundary conditions are

$$\lim_{S \downarrow 0} P_{ss}(S, t) = \lim_{S \uparrow \infty} P_{ss}(S, t) = 0.$$

Domain Extension in the Maturity Direction and Stationarity

To derive a forward FBP for American put values, we extend the domain to all $T \in [0, \bar{T}]$, keeping the strike fixed at K_0. Let $\pi(S,t;T)$ denote the American put value on this extended domain.

Now suppose stationarity, i.e. that $r(t)$, $q(t)$, $\sigma(S,t)$, and $\nu(x,S,t)$ are all independent of time t. Then theta is just the negative of the maturity derivative:

$$\frac{\partial}{\partial t}\pi(S,t;T) = -\frac{\partial}{\partial T}\pi(S,t;T)$$

Stationarity and Domain Extension in the Maturity Direction (cont'd)

The following relation holds in the extended domain:

$$
\begin{aligned}
&-\frac{\partial \pi}{\partial T}(S,t;T) + \frac{\sigma^2(S)S^2}{2}\frac{\partial^2 \pi}{\partial S^2}(S,t;T) + (r-q)S\frac{\partial \pi}{\partial S}(S,t;T) - r\pi(S,t;T) \\
&\qquad + \int_{-\infty}^{+\infty}\left[\pi(Se^x,t;T) - \pi(S,t;T) - \frac{\partial \pi}{\partial s}(S,t;T)S(e^x-1)\right]\nu(x,S)dx \\
&+\mathbf{1}_{S<S(t;T)}\left\{rK_0 - qS - \int_{\ln(S(t;T)/S)}^{\infty}\left[\pi(Se^x,t;T) - (K_0 - Se^x)\right]\nu(x,S)dx\right\} = 0,
\end{aligned}
$$

We note that one can fix t at t_0 and just solve the above problem in the S, T plane if desired. In this case, the initial condition is

$$\pi(S,t_0;t_0) = \max(K_0 - S, 0),$$

and the boundary conditions are

$$\lim_{S\downarrow 0}\pi_{ss}(S,t_0;T) = \lim_{S\uparrow\infty}\pi_{ss}(S,t_0;T) = 0.$$

Domain Extension in the Strike Direction

In the last slide, we assumed that the strike K was fixed at K_0 and that $r(t)$, $q(t)$, $\sigma(S,t)$, and $\nu(x,S,t)$ are all independent of time t. To derive a new PIDE for American put values, we further extend the domain of the problem to all $K > 0$. We also restore the dependence on t.

The backward PIDE holding on the three dimensional domain holds on the larger four dimensional domain with K_0 replaced by all $K > 0$.

On this larger domain, let $S(t;T,K)$ be the function relating the exercise surface to t, T, and K.

Additivity

Now assume that the log price has independent increments, i.e. is an additive process. Hence, the local volatility $\sigma(S,t)$ and the jump arrival rate $\nu(x,S,t)$ are both independent of the stock price S.

Then for each fixed t and T, the critical stock price $S(t;T,K)$ is proportional to K. It can be shown that for each fixed S, t, and T

$$S > S(t;T,K) \Rightarrow K < K(S,t;T)$$

where $K(S,t;T)$ relates the *critical strike price* to S,t and T. By definition, the critical strike price is the lowest strike price at which an American put is exercised early for fixed S,t,T.

Additivity and Domain Extension in the Strike Direction (cont'd)

The additivity of the log price process implies that the put value function $P(S, t; K, T)$ is linearly homogeneous in S and K. By Euler's theorem:

$$P(S, t; K, T) = S \frac{\partial}{\partial s} P(S, t; K, T) + K \frac{\partial}{\partial K} P(S, t; K, T).$$

Differentiation w.r.t. S and K establishes that:

$$S^2 \frac{\partial^2}{\partial s^2} P(S, t; K, T) = K^2 \frac{\partial^2}{\partial K^2} P(S, t; K, T).$$

Additivity and Domain Extension in the Strike Direction (cont'd)

After substitution and some straightforward calculations, we obtain the following hybrid relation:

$$\frac{\partial P(S,t;K,T)}{\partial t} + \frac{\sigma^2(t)K^2}{2}\frac{\partial^2 P(S,t;K,T)}{\partial K^2} - [r(t) - q(t)]\,K\frac{\partial P(S,t;K,T)}{\partial K} - q(t)P(S,t;K,T)$$
$$+\int_{-\infty}^{+\infty}\left[P(S,t;Ke^{-x},T) - P(S,t;K,T) - \frac{\partial P(S,t;K,T)}{\partial K}K(e^{-x}-1)\right]e^{x}\nu(x,t)dx$$
$$+\mathbf{1}_{K>\bar{K}(S,t;T)}\left\{r(t)K - q(t)S - \int_{\ln(K/\bar{K}(S,t:T))}^{\infty}\left[P(S,t;Ke^{-x},T) - (Ke^{-x}-S)\right]e^{x}\nu(x,t)dx\right\}$$
$$= 0.$$

We note that one can fix S and T at say S_0 and T_0 and just solve the PIDE in the K,t plane if desired.

Additivity and Domain Extension in the Strike Direction (cont'd)

In this case, the terminal condition is:

$$P(S_0, T_0; K, T_0) = \max(K - S_0, 0),$$

and the boundary conditions are

$$\lim_{K \downarrow 0} P_{KK}(S_0, t; K, T_0) = \lim_{K \uparrow \infty} P_{KK}(S_0, t; K, T_0) = 0.$$

Note that eliminating jumps reduces the PIDE to a PDE arising in the special case of the time-dependent Black Scholes model.

The Forward Free Boundary Problem

We now assume that we have both stationarity and additivity. In other words, the log price is a *Lévy* process and $r(t)$, $q(t)$, $\sigma(S,t)$, and $\nu(x,S,t)$ are all independent of both time t and the stock price S.

Stationarity implies that the put value function $P(S,t;K,T)$ depends on t and T only through $T-t$. It again follows that:

$$\frac{\partial}{\partial t}P(S,t;K,T) = -\frac{\partial}{\partial T}P(S,t;K,T)$$

The Forward Free Boundary Problem (cont'd)

Substituting in the hybrid relation yields the following:

$$\frac{\partial P(S,t;K,T)}{\partial T} - \frac{\sigma^2 K^2}{2}\frac{\partial^2 P(S,t;K,T)}{\partial K^2} + (r-q)K\frac{\partial P(S,t;K,T)}{\partial K} + qP(S,t;K,T)$$
$$-\int_{-\infty}^{+\infty}\left[P(S,t;Ke^{-x},T) - P(S,t;K,T) - \frac{\partial P(S,t;K,T)}{\partial K}K(e^{-x}-1)\right]e^{x}\nu(x)dx$$
$$-\mathbf{1}_{K>\bar{K}(s,t:T)}\left\{rK - qS - \int_{\ln(K/\bar{K}(S,t;T))}^{\infty}\left[P(S,t;Ke^{-x},T) - (Ke^{-x}-S)\right]e^{x}\nu(x)dx\right\} = 0,$$

where $\nu(x)$ is the *Lévy density*.

The Forward Free Boundary Problem (cont'd)

We note that one can fix S and t at say S_0 and t_0 and just solve the forward PIDE in the K,T plane if desired. In this case, the initial condition is:

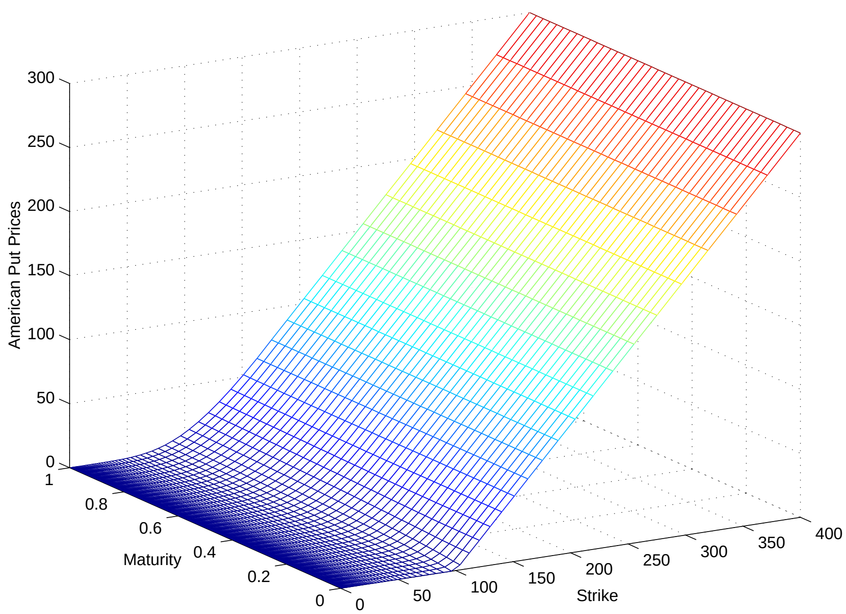
$$P(S_0, t_0; K, t_0) = \max(K - S_0, 0),$$

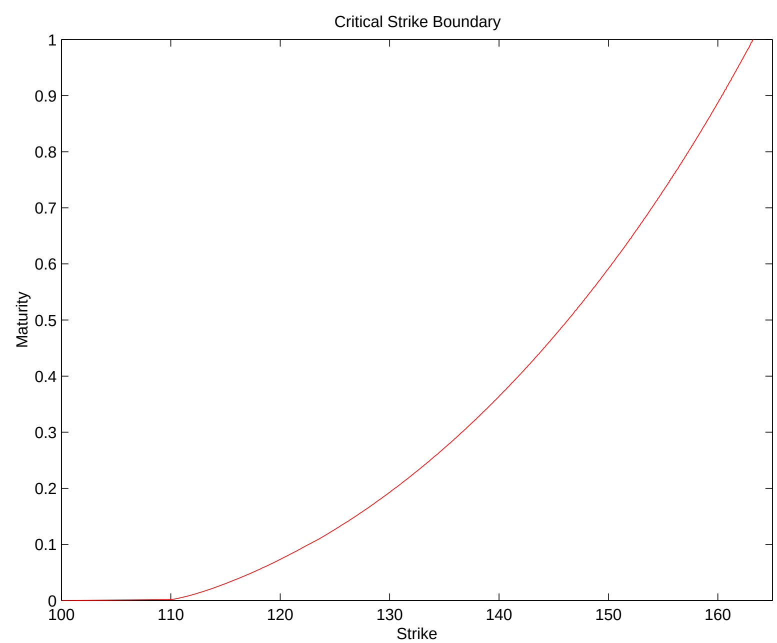
and the boundary conditions are

$$\lim_{K \downarrow 0} P_{KK}(S_0, t_0; K, T) = \lim_{K \uparrow \infty} P_{KK}(S_0, t_0; K, T) = 0.$$

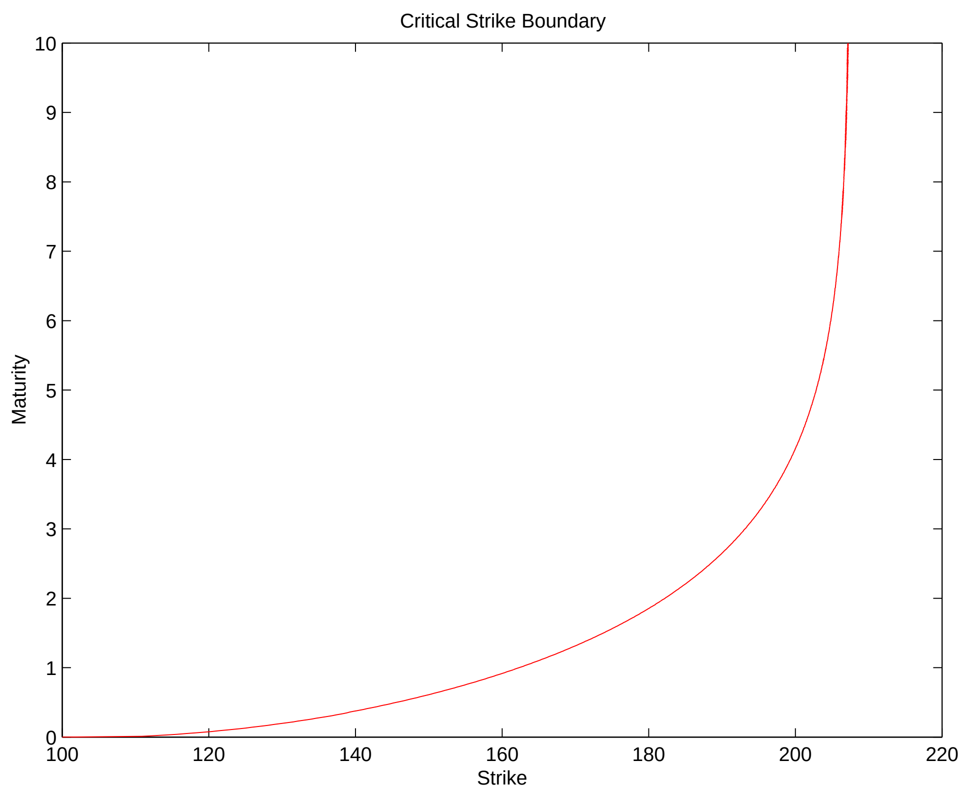
American Put Prices and Critical Strike Boundary

In this example, the variables are: spot S_0=100, risk-free rate $r = 0.05$, dividend rate $q = .02$, volatility $\hat{\sigma} = .20$, and VG parameters $\sigma = .3$, $\nu = .25$, $\theta = -.3$.





Asymptotic Behavior of the Critical Strike Boundary



Future Research

Forward PIDE for Double Knockouts with arrival rate depending on spot

Forward P(I)DE for American options with local volatility surface

Forward P(I)DE for European options under stochastic volatility

The evolution of the first item is deterministic, while the last two are much more stochastic!